Two-neighbour stochastic cellular automata and their planar lattice duals

1995 J. Phys. A: Math. Gen. 28 3935

(http://iopscience.iop.org/0305-4470/28/14/014)

J. Phys. A: Math. Gen. 28 (1995) 3935–3957. Printed in the UK

Two-neighbour stochastic cellular automata and their planar lattice duals

Makoto Katori and Hiroshi Tsukahara

Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112, Japan

Received 28 March 1995

Abstract. Two-neighbour stochastic cellular automata (SCA) are the set of one-dimensional discrete-time interacting particle systems with two parameters, which show non-equilibrium phase transitions from the extinction phase to the survival phase. The phase diagram was first studied by Kinzel using a numerical method called transfer-matrix scaling. For some parameter region the processes can be defined as directed percolation models on the spatio-temporal plane and the bond- and site-directed percolation models are included as special cases. Extending the argument of Dhar, Barma and Phani originally given for bond-directed percolation, we introduce diode–resistor percolation models which are the planar lattice duals of the SCA and give rigorous lower bounds for the critical line. In special cases, our results give $0.6885 \leqslant \alpha_c$ and $0.6261 \leqslant \beta_c$, where α_c and β_c denote the critical probabilities of the site- and bond-directed percolation models on the square lattice, respectively. Combining the upper bound for the critical line recently proved by Liggett, we summarize the rigorous results for the phase diagram of the systems. Results of computer simulation are also shown.

1. Introduction

Stochastic cellular automata (SCA) are interacting particle systems in discrete space and time, in which the evolution satisfies local stochastic rules. Since we expect that they provide simple mathematical models for systems which are far from thermal equilibrium (Dickman 1993), the stochastic rules are *not* assumed to be given by the local Boltzmann weight *nor* to satisfy detailed balance. Kinzel (1985) introduced some elementary one-dimensional SCA which have absorbing states as trivial stationary states. Continuous phase transitions between the absorbing states and the *active* stationary states are observed even in one-dimensional SCA, in contrast to systems satisfying detailed balance. Kinzel studied the phase diagrams and critical phenomena using transfer-matrix scaling.

In the present paper, we revisit a set of the SCA of Kinzel, the two-state SCA with two neighbours, and give some rigorous results for the phase diagrams. Each site x on a one-dimensional integer lattice takes one of two states, 0 (vacancy) and 1 (particle), and the value of site x at time $t+1$, which we write as $\eta_{t+1}(x)$, is determined to satisfy the following stochastic rules. The probability for $\eta_{t+1}(x) = 1$ is zero if $\eta_t(x-1) = \eta_t(x+1) = 0$ (no particle in the neighbourhood), p_1 if $\eta_t(x-1) = 1, \eta_t(x+1) = 0$ or $\eta_t(x-1) = 0, \eta_t(x+1) = 1$ (one particle in the neighbourhood), and p_2 if $\eta_t(x-1) = \eta_t(x+1) = 1$ (two particles in the neighbourhood). If we regard the state $\eta_t(x) = 1$ (resp. $\eta_t(x) = 0$) as representing an infected (resp. a healthy) individual at x, this SCA can be considered as a simple model of the spread of infection of a disease, whose continuous-time version is known as the contact process (Harris 1974, Liggett 1985). If p_1 and p_2 are small, the disease becomes extinct with probability one (*extinction phase*), while if p_1 and p_2 are sufficiently large, there is a positive probability for which the infection process will survive for all time (*survival phase*).

One-dimensional SCA can be defined as two-dimensional statistical models on the spatio-temporal plane. Durrett (1988) called the above SCA of Kinzel two-neighbour systems and showed that they are constructed as generalized directed percolation models on the spatio-temporal plane when the parameters satisfy the relation

$$p_1 \leqslant p_2 \leqslant 2p_1 . \tag{1.1}$$

From this point of view, the survival phase and the extinction phase are identified with the percolation phase and the non-percolation phase, respectively. We have a critical line between these two phases on the (p_1, p_2)-phase diagram. It should be noted that the two-neighbour SCA with (1.1) includes bond- and site-directed percolation as special cases. The intersection of the critical line and the line $p_2 = p_1$ (resp. the line $p_2 = p_1(2-p_1)$) gives the critical probability α_c (resp. β_c) of the site (resp. bond) directed percolation.

Recently Liggett (1994) proved the upper bound of the critical line for the systems in the parameter region $\frac{1}{2} \leqslant p_1 \leqslant p_2 \leqslant 1$. As a special case, his bound gives the best upper bound for β_c: $\beta_c \leqslant \frac{2}{3}$. For the site model, his results give $\alpha_c \leqslant \frac{3}{4}$, which is slightly larger than the best upper bound 0.7491 given by Balister *et al* (1994). On the other hand, Dhar (1982) invented a series of strategies which improve lower bounds for the bond-directed percolation successively. His second strategy gives $0.6261 \leqslant \beta_c$ and the third one gives the best lower bound, $0.6298 \leqslant \beta_c$. These lower bounds are excellent, since the most reliable value known so far is $\beta_c = 0.644\,701 \pm 0.000\,001$ estimated by Baxter and Guttmann (1988). The difference is less than 3%. However, corresponding good lower bounds have not yet been given for the site-directed percolation and other cases.

In this paper we show that Dhar's method can be extended for the two-neighbour SCA if the condition (1.1) is satisfied, and give lower bounds for the critical line. Our best bound is reduced to the result given by the second strategy of Dhar, for the bond-directed percolation limit. For the site case, our result gives $0.6885 \leqslant \alpha_c$. The difference from the value estimated by Onody and Neves (1992), $\alpha_c = 0.705\,489 \pm 0.000\,004$, is only 2.4%. Combining Liggett's upper bound and ours, we present figures which summarize the rigorous bounds for the critical line, and discuss the phase diagram of the two-neighbour SCA. Dhar's method is based on the observation by Dhar *et al* (1981) that bond-directed percolation is the planar lattice dual of the diode–resistor percolation model. Dhar *et al* (1981) introduced an angle ϕ which characterizes the asymptotic behaviour of the spread of the diode–resistor percolation region. However, the definition of this angle is not entirely clear in their short paper. Therefore we examine their argument carefully, comparing it with Durrett's approach for the edge process (Durrett 1984), and give a precise definition for ϕ. The argument presented in this paper is necessary for giving mathematical justification to Dhar's method. Another verification of the Dhar–Barma–Phani argument was given by Wierman (1983). Though both proofs use the same mathematical technique and the subadditive ergodic theorem, construction is different from each other (see remark 3.3).

The paper is organized as follows. In section 2 we define the two-neighbour SCA based on the percolation substructure and briefly explain time-reversal duality, which gives a useful characterization of the critical values. Section 3 is devoted to reformulating the planar lattice duality first observed by Dhar *et al* (1981). Dhar's method for the lower bounds is extended in section 4 and the phase diagram of the system is discussed in section 5. Details of the calculation are found in the appendix. Numerical data obtained by computer simulation are also shown in section 5. At the end of section 5 some remarks on future problems are given.

2. Two-neighbour stochastic cellular automata and time-reversal duality

2.1. The process

We consider the set of discrete time processes η_t on a spatio-temporal plane

$$V = \{(x,t) \in \mathbf{Z}^2 : x+t = \text{even},\ t = 0,1,2,\ldots\}. \tag{2.1}$$

They are the two-valued stochastic cellular automata (discrete-time interacting particle systems); each site x takes one of the two states 0 (vacant) and 1 (occupied by a particle). The state at time t is thus given as a set η_t of sites occupied by particles. Let $\mathbf{Z}_e = \{\ldots,-4,-2,0,2,4,\ldots\}$ and $\mathbf{Z}_o = \{\ldots,-3,-1,1,3,\ldots\}$. Then $\eta_t \subset \mathbf{Z}_e$ for t = even and $\eta_t \subset \mathbf{Z}_o$ for t = odd. The time evolution is given by

$$P(x \in \eta_{t+1} \mid \eta_t) = f(|\eta_t \cap \{x-1, x+1\}|) \tag{2.2}$$

where $|A|$ denotes the number of sites included in a set A. That is, it is a Markov process and the state of the site x at time $t+1$ depends only on the states of its two neighbouring sites at time t.

Kinzel (1985) introduced the set of two-neighbour systems which is parameterized by p_1 and p_2 as

$$f(0) = 0 \qquad f(1) = p_1 \qquad f(2) = p_2 \tag{2.3}$$

with $0 \leqslant p_1 \leqslant 1, 0 \leqslant p_2 \leqslant 1$. Figure 1 shows the elementary processes.

As explained clearly in section 5b of Durrett (1988), the processes can be identified with directed percolation models on the spatio-temporal plane V. At each triangle of three points $(x-1,t)$, $(x+1,t)$ and $(x,t+1)$, we place the following 'gadgets' independently. Two arrows with probability p, one arrow with probability q (the left and right ones, respectively) and nothing with probability $1-p-2q$ (see figure 2). The arrow from $(x-1,t)$ to $(x,t+1)$ or that from $(x+1,t)$ to $(x,t+1)$ denotes that the directed bond is open. We say 'there is an open path from (x,t) to $(y,t+n)$ $(n \geqslant 1)$' if there is a sequence $(x_0,t) = (x,t), (x_1,t+1), \ldots, (x_n,t+n) = (y,t+n)$ of points in V such that for each $0 \leqslant i \leqslant n-1$ the bond from $(x_i,t+i)$ to $(x_{i+1},t+i+1)$ is open, and write $(x,t) \longrightarrow (y,t+n)$ for short. If we define a process by setting

$$\eta_t^x = \{y : (x,0) \longrightarrow (y,t)\} \tag{2.4}$$

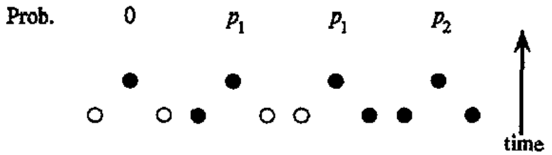

Figure 1. The elementary processes of the two-neighbour SCA. The full (resp. open) circles denote particles (resp. vacancies).

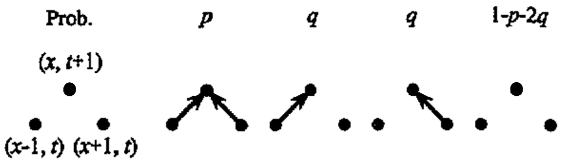

Figure 2. The gadgets for constructing the two-neighbour SCA η_t on the spatio-temporal plane V.

and

$$\eta_t^A = \bigcup_{x \in A} \eta_t^x \tag{2.5}$$

then the resulting system is nothing but the process (2.2) with (2.3) starting from the state A with

$$p_1 = p + q \qquad \text{and} \qquad p_2 = p + 2q\,. \tag{2.6}$$

Since we must have $0 \leqslant p \leqslant 1, 0 \leqslant q \leqslant 1$ and $0 \leqslant 1 - p - 2q \leqslant 1$, the above identification with the directed percolation is possible if and only if

$$p_1 \leqslant p_2 \leqslant 2p_1\,. \tag{2.7}$$

Since we will use the above percolation substructure throughout this paper, we assume (2.7) from now on.

2.2. *Special cases*

It is easy to confirm that the above set of processes includes the well known percolation models as special cases if the parameters are chosen as follows.

Site-directed percolation with site concentration α

$$p_1 = \alpha \qquad p_2 = \alpha\,. \tag{2.8}$$

Bond-directed percolation with bond concentration β

$$p_1 = \beta \qquad p_2 = \beta(2 - \beta)\,. \tag{2.9}$$

Mixed site-bond directed percolation with the site (resp. bond) concentration α (resp. β)

$$p_1 = \alpha\beta \qquad p_2 = \alpha\beta(2 - \beta)\,. \tag{2.10}$$

2.3. *Critical line*

Since the spontaneous creation of particles is forbidden, the empty state $\eta_t = \emptyset$ is absorbing. When $p_2 \geqslant p_1$, the population at time t is a non-decreasing function of the population at time zero. This property is called *attractiveness* and it follows that the process starting from the state with all sites occupied by particles, $\eta_t^{\mathbf{Z}_e}$, will have the highest probability for survival. If $\lim_{t\to\infty} P(\eta_t^{\mathbf{Z}_e} \neq \emptyset) = 0$, then any process will become extinct with probability one. On the other hand, in some parameter region in (2.7) we have $P(\eta_t^{\mathbf{Z}_e} \neq \emptyset$ for all $t \geqslant 0) > 0$. It is verified for $p_2 \geqslant p_1$ that this probability is a monotonically non-decreasing function of p_1 and p_2. Then the two-parameter space (2.7) will be divided into two regions; the region where $\lim_{t\to\infty} P(\eta_t^{\mathbf{Z}_e} \neq \emptyset) = 0$ and the region where $P(\eta_t^{\mathbf{Z}_e} \neq \emptyset$ for all $t \geqslant 0) > 0$. We will call the former the *extinction phase* and the latter the *survival phase*. The boundary between these two phases will be called the critical line, since critical phenomena are observed on this line.

2.4. *Time-reversal dual process*

It is difficult to estimate the probability $P(\eta_t^{\mathbf{Z}_e} \neq \emptyset$ for all $t \geqslant 0)$, since it is defined for the process $\eta_t^{\mathbf{Z}_e}$, whose particle number is infinite. As explained below, however, this quantity turns out to be equal to the probability for a process in which the number of particles remains finite if $t < \infty$.

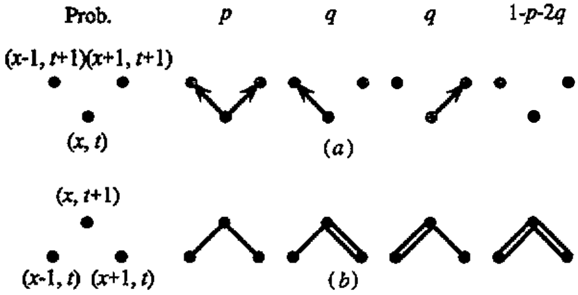

Figure 3. (*a*) The gadgets for constructing the time-reversal dual process $\tilde{\eta}_t$. They are obtained from the gadgets in figure 2 by reversing the time direction. (*b*) For each gadget in (*a*), we assign a new gadget which consists of resistors (double lines) and/or diodes (single lines). These gadgets are placed on triangles of three points on the lattice V^* which is the planar dual of V.

In subsection 2.1, we showed that the process η_t can be constructed as a union of percolation paths on the spatio-temporal plane V by using the gadgets shown in figure 2. In the same way, we can define another stochastic process $\tilde{\eta}_t$ by using the time-reversed gadgets shown in figure 3(*a*). This process $\tilde{\eta}_t$ is said to be a time-reversal dual process of η_t and by definition we can conclude the following relation:

$$\{\eta_t^A \cap B \neq \emptyset\} = \{A \cap \tilde{\eta}_t^B \neq \emptyset\} \qquad \text{with probability one} \tag{2.11}$$

for all sets A and B. This relation is called the *time-reversal duality relation* (Durrett 1984) or specifically the *coalescing duality relation* to distinguish it from other kinds of time-reversal duality relations. For more details, see Durrett (1988) and Inui *et al.* (1995). Let $A = \mathbf{Z}_e$, $B = \{0\}$ for $t =$ even in (2.11). Then we have

$$P(\{0\} \in \eta_t^{\mathbf{Z}_e}) = P(\tilde{\eta}_t^{\{0\}} \neq \emptyset) \qquad \text{for} \quad t = \text{even}. \tag{2.12}$$

Since the empty set $\emptyset$ is a single absorbing state of the process $\tilde{\eta}_t^{\{0\}}$, the RHS of (2.12) is decreasing in t and has the limit

$$\rho(p, q) = \lim_{t\to\infty} P(\tilde{\eta}_t^{\{0\}} \neq \emptyset) \tag{2.13}$$

which is a function of the parameters p, q with

$$0 \leqslant q \leqslant \tfrac{1}{2} \qquad 0 \leqslant p \leqslant 1 - 2q\,. \tag{2.14}$$

By the duality relation (2.12), we have

$$\rho(p, q) > 0 \Longleftrightarrow \text{survival phase}$$
$$\rho(p, q) = 0 \Longleftrightarrow \text{extinction phase}.$$

This quantity $\rho(p, q)$ is a non-decreasing function of p and q if (2.14) and the precise definition of the critical line on the (p, q)-plane is given as follows.
For $0 \leqslant q \leqslant \frac{1}{2}$,

$$\begin{aligned} p_c(q) &= \sup\{p : \rho(p, q) = 0\} \\ &= \inf\{p : \rho(p, q) > 0\}. \end{aligned} \tag{2.15}$$

This line will be mapped to the critical line $p_1 = p_{1c}(p_2)$ on the (p_1, p_2)-plane by the relation (2.6).

3. Edge process and diode–resistor percolation

3.1. Edge speed $\alpha(p, q)$

Following the argument of Durrett (1984) for two-dimensional directed percolation, we can give another characterization of $p_c(q)$. Consider the right edge process $\tilde{\eta}_t^{\mathbf{Z}_-}$, whose initial state is $\mathbf{Z}_- = \{\ldots, -6, -4, -2, 0\}$ and define the right edge as

$$r_t = \sup\{x : x \in \tilde{\eta}_t^{\mathbf{Z}_-}\}. \tag{3.1}$$

Intuitively speaking, the behaviour of this right edge r_t for $t \to \infty$ will characterize $p_c(q)$. That is, if $r_t \to \infty$ as $t \to \infty$ the process will survive, while if $r_t \to -\infty$ as $t \to \infty$ the process will die out. Durrett (1984) gave rigorous meaning for this observation. The simple extension of his theorem to the present generalized process $\tilde{\eta}_t$ will be given below.

For $0 \leqslant q \leqslant \frac{1}{2}, 0 \leqslant p \leqslant 1-2q$,

$$\lim_{t\to\infty} \frac{r_t}{t} = \alpha(p,q) \text{ exists almost surely} \tag{3.2}$$

where $\alpha(p,q) = \inf_{t>0} E[r_t]/t$.

The quantity $\alpha(p,q)$ is the edge speed and characterizes the critical values as

$$\begin{aligned} p_c(q) &= \inf\{p : \alpha(p,q) > 0\} \\ &= \sup\{p : \alpha(p,q) < 0\}. \end{aligned} \tag{3.3}$$

3.2. Dhar–Barma–Phani's characterization of edge processes

Dhar *et al* (1981) proposed another characterization of the asymptotic behaviour of the edge processes. First we explain the basic idea of their argument using some figures. The precise description will be given in the next subsection.

In figure 4(a) a typical realization of the edge process is shown. Dhar *et al* (1981) concentrated on the 'contour' of the right edge, which is indicated by a wavy line. We will

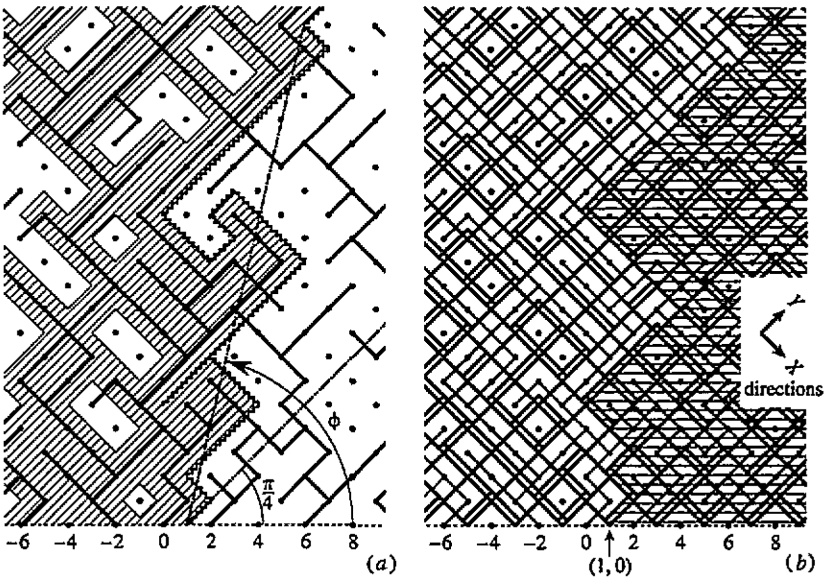

Figure 4. (a) A typical realization of the right edge process $\tilde{\eta}_t^{\mathbf{Z}_-}$. The spatio-temporal region which is occupied by particle is hatched. The wavy line indicates the 'contour' of the right edge. The angle between the contour and the line $t = 0$ is denoted by ϕ. (b) The diode–resistor system on V^* corresponding to the realization (a), where diodes (resp. resistors) are denoted by single (resp. double) lines. The orientation of diodes is in the direction X or Y. The shaded region is the wet region percolated from the point (1, 0).

define the (asymptotic) angle ϕ between this contour and the line $t = 0$, which is just the half of the angle denoted by θ and called the wedge angle in Dhar *et al* (1981). This angle $\phi \in [\pi/4, \pi]$ and we observe that

$$\begin{aligned} &\frac{\pi}{4} \leqslant \phi < \frac{\pi}{2} \Longleftrightarrow \text{survival} \\ &\phi > \frac{\pi}{2} \Longleftrightarrow \text{extinction}. \end{aligned} \tag{3.4}$$

It should be noticed that this contour of the edge is defined on bonds not in the lattice V but in its planar dual lattice

$$V^* = \{(x, t) \in \mathbf{Z}^2 : x + t = \text{odd}, t = 0, 1, 2, \ldots\}. \tag{3.5}$$

In order to indicate the contour of edge process for bond-directed percolation, Dhar *et al* (1981) introduced the diode–resistor percolation model on the dual lattice V^*. The similar procedure can also be done for the present process $\tilde{\eta}_t^{\mathbf{Z}_-}$ as follows. For each gadget of $\tilde{\eta}_t$ in figure 3(a), we assign a new gadget as shown in figure 3(b). Here double lines represent resistors (two-way conducting bonds) and single lines represent diodes (one-way conducting bonds). Figure 4(b) shows the diode–resistor system on V^* corresponding to the realization shown in figure 4(a). We assume that the orientation of diodes is in the direction $(1, 1)$ or $(1, -1)$; current is allowed only in these directions through diodes. Then we put a source of fluid at the point $(1, 0)$ and consider the percolation problem on this diode–resistor system. The lattice V^* will be divided into two regions; the wet region percolated from $(1, 0)$ and the dry region. The boundary of the wet region behaves in a similar way to the contour of the original edge process. We give a precise definition for ϕ in the next subsection.

3.3. *Precise definition of the angle ϕ*

We consider the diode–resistor percolation model on V^*. First we change the coordinates for convenience. We shift the coordinates so that $(1, 0) \in V^*$ becomes a new origin and rotate the axis by $\pi/4$ and rescale the units. That is; $(x, t) \in V^* \mapsto (X, Y) \in \mathbf{Z}^2$ as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x - 1 \\ t \end{pmatrix}. \tag{3.6}$$

At each pair of bonds connecting (X, Y) with $(X + 1, Y)$ and with $(X, Y - 1)$ on this new plane, we place independently one of the four gadgets shown in figure 5 with the appropriate probability. The orientation of diodes is now pre-assigned to be in the direction of increasing X or Y coordinates. Since we have considered the region $t \geqslant 0$ in V^*, we will consider the half of $\mathbf{Z}^2$,

$$U = \{(X, Y) \in \mathbf{Z}^2 : Y \geqslant X\}. \tag{3.7}$$

Figure 6(a) shows the contour of the edge process which was shown in figure 4 in this new coordinates.

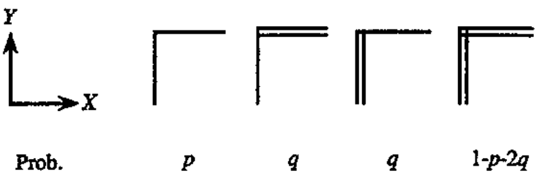

Figure 5. Four kinds of gadgets for the present diode–resistor system. At each pair of bonds on $\mathbf{Z}^2$, we place one of them independently with the appropriate probability.

We introduce the following notation for sets of bonds:

$$
\begin{aligned}
v_i &= \text{the set of vertical bonds in the column } X = i, \\
h_i &= \text{the set of horizontal bonds between the columns } X = i \text{ and } X = i+1
\end{aligned}
\tag{3.8}
$$

and

$$B_i = \left(\bigcup_{k\geqslant i} v_k\right) \bigcup \left(\bigcup_{k\geqslant i} h_k\right). \tag{3.9}$$

We will write the set of bonds which are the elements of B_i and both of whose edges are in U as $B_i \cap U$. In this diode–resistor percolation, if there is a path from (X_1, Y_1) to (X_2, Y_2) such that all the bonds consisting the path are included in a set C, then we write

$$(X_1, Y_1) \xrightarrow[C]{*} (X_2, Y_2). \tag{3.10}$$

For each realization of the diode–resistor system, we can define the following points $P_i = (-i, Y_i),\ i = 0, 1, 2, \ldots$ as

$$
\begin{aligned}
Y_0 &= 0 \\
Y_i &= \min\{Y : (-i, Y) \in U \text{ and } (0,0) \xrightarrow[B_{-i}\cap U]{*} (-i, Y)\} \qquad \text{for} \quad i = 1, 2, \ldots.
\end{aligned}
\tag{3.11}
$$

See figure 6(a), for example. If we let

$$U(i) = \{(X, Y) \in \mathbf{Z}^2 : Y - Y_i \geqslant X + i\} \qquad \text{for} \quad i = 0, 1, 2, \ldots \tag{3.12}$$

and define for $0 \leqslant j \leqslant i$

$$Y_{j,i} = \min\{Y : (-i, Y) \in U(j) \qquad \text{and} \qquad (-j, Y_j) \xrightarrow[B_{-i}\cap U]{*} (-i, Y)\} - Y_j \tag{3.13}$$

then $U(0) = U$ and $Y_{0,i} = Y_i$. By these definitions, we can observe that $Y_{j,i}$ is *subadditive* in the sense (see figure 6(b))

$$Y_{0,i} \leqslant Y_{0,j} + Y_{j,i} \qquad \text{for} \quad 0 \leqslant j \leqslant i. \tag{3.14}$$

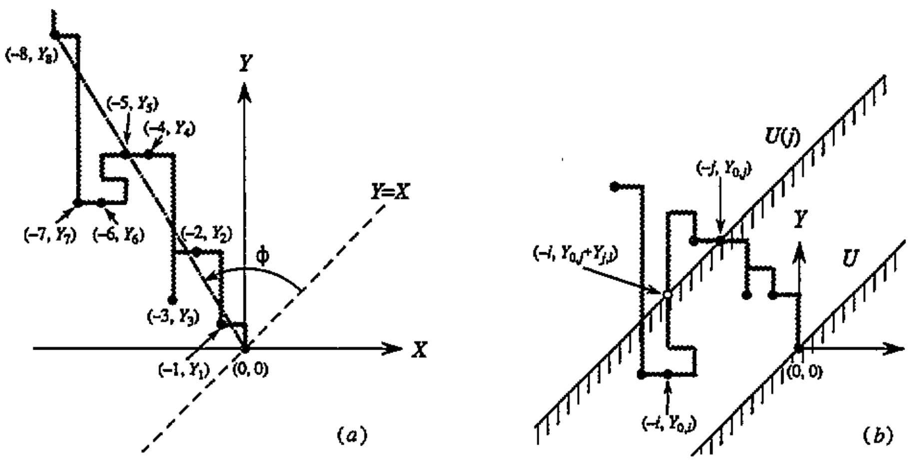

Figure 6. (a) The contour of the edge process in figure 4(a) is shown in the (X, Y) plane. Following (3.11) and (3.12), the points $P_i = (-i, Y_i),\ i = 0, 1, 2, \ldots$ are determined as shown for this realization. (b) Since the point $(-i, Y_{0,j} + Y_{j,i})$ is defined in the region $U(j)$, its ordinate is equal to or greater than $Y_{0,i}$.

We also find that $-n \leqslant E[Y_n] < \infty$ for any n unless $p = 1$. We can show that the subadditive ergodic theorem (see p 277 in Liggett 1985) is applicable to $Y_{j,i}$ and have the following lemma.

Lemma 3.1. Let

$$y_n = E[Y_n]. \tag{3.15}$$

Then

$$\exists y = \lim_{n\to\infty} \frac{y_n}{n} = \inf_{n\geqslant 1} \frac{y_n}{n} \in [-1, \infty) \tag{3.16}$$

and

$$Y_\infty = \lim_{n\to\infty} \frac{Y_n}{n} \qquad \text{exists almost surely with} \quad -1 \leqslant Y_\infty < \infty \tag{3.17}$$

with

$$E[Y_\infty] = y. \tag{3.18}$$

By using this lemma, we can give a definition of ϕ.

Definition 3.2.

$$\cot\left(\phi - \frac{\pi}{4}\right) = y. \tag{3.19}$$

The criterion (3.4) is rewritten as follows:

$$\begin{aligned} y > 1 &\Longleftrightarrow \text{survival} \\ y < 1 &\Longleftrightarrow \text{extinction}. \end{aligned} \tag{3.20}$$

Remark 3.3. In this paper we consider the diode–resistor percolation on a half plane (3.7), since we concentrate on the asymptotic behaviour of edge processes for characterizing the critical values. On the other hand, Wierman (1983) studied the model on a whole plane and proved the well-definiteness of the wedge angle θ which is just the twice of ϕ.

4. Approximate processes and lower bounds for $p_c(q)$

4.1. Approximate procedure

Let τ_{-n} be the translation operator which shifts a region to the left by n columns. By the definition (3.9), we see

$$B_{-n} = \tau_{-(n-1)} B_{-1} \qquad n = 2, 3, 4, \ldots. \tag{4.1}$$

We find that the sequence $\{Y_n\}_{n=0,1,2,\ldots}$ can be given successively as follows:

$$\begin{aligned} &Y_0 = 0 \\ &Y_{n+1} = \min\{Y : Y \geqslant -(n+1) \text{ and } (-n, Y_n) \xrightarrow[\tau_{-n}B_{-1}\cap U]{*} (-(n+1), Y)\} \\ &\qquad\qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{4.2}$$

The sequence $\{Y_n\}_{n=0,1,2,\ldots}$ can be regarded as a process, but it is not a Markov process.

Dhar (1982) proposed a method to approximate this non-Markov process. His strategy is to introduce an approximate process $\{Y_n^C\}$ which is defined successively in a similar way to (4.2), but for which the set B_{-1} is replaced by its subset

$$C \subset B_{-1}. \tag{4.3}$$

By this definition, we have

$$Y_n^C \geqslant Y_n \qquad n = 0, 1, 2, \ldots \tag{4.4}$$

since the percolation paths which we can consider are more restricted. As for Y_n, we can define the limit

$$y^C = \lim_{n\to\infty} \frac{1}{n} E[Y_n^C] \tag{4.5}$$

and it follows that

$$y^C \geqslant y . \tag{4.6}$$

In order to calculate y^C, we consider another process $\{\bar{Y}_n^C\}$, which is defined as follows:

$$\begin{aligned} &\bar{Y}_0 = 0 \\ &\bar{Y}_{n+1} = \inf\{Y : (-n, \bar{Y}_n) \xrightarrow[\tau_{-n}C]{*} (-(n+1), Y)\} \qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.7}$$

It should be noted that in this definition the restriction for the paths so that they are in the region U is omitted and thus

$$\{\bar{Y}_{n+1}^C - \bar{Y}_n^C\} \stackrel{d}{=} \{\bar{Y}_{n+k+1}^C - \bar{Y}_{n+k}^C\} \qquad \text{for any} \quad k \geqslant 1 . \tag{4.8}$$

Therefore we can define a random variable W^c which equals $\{\bar{Y}_{n+1}^C - \bar{Y}_n^C\}$ in distribution and

$$\lim_{n\to\infty} \frac{1}{n} E[\bar{Y}_n^C] = E[W^C] . \tag{4.9}$$

If we write the maximum of the values a and b as $a \vee b$, we can conclude that the limit (4.5) is given by

$$y^C = E[W^C] \vee -1 . \tag{4.10}$$

Combining the above observation and the criterion (3.20) gives the following useful lemma.

Lemma 4.1.

$$E[W^C] < 1 \Longrightarrow \text{extinction} . \tag{4.11}$$

4.2. *Calculation*

In this subsection we show an outline of the calculation of $E[W^C]$ for the choice

$$C = v_0 \cup h_{-1} \cup v_{-1} \tag{4.12}$$

or

$$\tau_{-n}C = v_{-n} \cup h_{-(n+1)} \cup v_{-(n+1)} \qquad n = 0, 1, 2, \ldots . \tag{4.13}$$

The details will be given in the appendix.

Assume that $\bar{Y}_n^C$ is obtained as the ordinate of the lowest point in the column $X = -n$ such that there is a path from the point $(-(n-1), \bar{Y}_{n-1}^C)$ in the set of bonds $\tau_{-(n-1)}C$. Let

$$\bar{Y}_{n+1}^{(1)} = \inf\{Y : (-n, \bar{Y}_n^C) \xrightarrow[v_{-n}\cup h_{-(n+1)}]{*} (-(n+1), Y)\} \tag{4.14}$$

$$\bar{Y}_{n+1}^{(2)} = \inf\{Y : (-(n+1), \bar{Y}_{n+1}^{(1)}) \xrightarrow[v_{-(n+1)}]{*} (-(n+1), Y)\} . \tag{4.15}$$

Then by the definition (4.7) with (4.13), we find

$$\bar{Y}_{n+1}^C = \inf\{Y : (-(n+1), \bar{Y}_{n+1}^{(2)}) \xrightarrow[v_{-n}\cup h_{-(n+1)}\cup v_{-(n+1)}]{*} (-(n+1), Y)\}. \tag{4.16}$$

We define

$$r_n = \bar{Y}^{(1)}_{n+1} - \bar{Y}^C_n \tag{4.17}$$

$$s_n = \bar{Y}^{(1)}_{n+1} - \bar{Y}^{(2)}_{n+1} \tag{4.18}$$

$$t_n = \bar{Y}^{(2)}_{n+1} - \bar{Y}^C_{n+1}\,. \tag{4.19}$$

See figure 7, for example.

By definition r_n and s_n are determined by the bond configuration on $v_{-(n+1)} \cup h_{-(n+1)}$ and are thus independent of $\{(r_k, s_k, t_k) : k \leqslant n-1\}$. In some cases, however, t_n depends not only on r_n and s_n but also on r_{n-1} and s_{n-1}. We consider the distribution function of the set of random variables $(r_n, s_n, t_n, r_{n-1}, s_{n-1})$, which we simply write as

$$P(r, s, t, \tilde{r}, \tilde{s}) = \mathrm{Prob}((r_n, s_n, t_n, r_{n-1}, s_{n-1}) = (r, s, t, \tilde{r}, \tilde{s})) \tag{4.20}$$

for $0 \leqslant r, s, t, \tilde{r}, \tilde{s} < \infty$. The expectation of the function of (r_n, s_n, t_n) is defined as

$$E[f(r_n, s_n, t_n)] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\tilde{r}=0}^{\infty}\sum_{\tilde{s}=0}^{\infty} f(r, s, t) P(r, s, t, \tilde{r}, \tilde{s}) \tag{4.21}$$

and we obtain

$$E[W^C] = E[r_n] - E[s_n] - E[t_n]\,. \tag{4.22}$$

The distribution (4.20) is rewritten as follows by using the appropriate conditional probabilities

$$\begin{aligned} P(r, s, t, \tilde{r}, \tilde{s}) &= P(t|r, s, \tilde{r}, \tilde{s}) P(r, s|\tilde{r}, \tilde{s}) P(\tilde{r}, \tilde{s}) \\ &= P(t|r, s, \tilde{r}, \tilde{s}) P(r, s) P(\tilde{r}, \tilde{s}) \end{aligned} \tag{4.23}$$

where $P(\omega_1|\omega_2)$ denotes the conditional probability of ω_1 given ω_2.

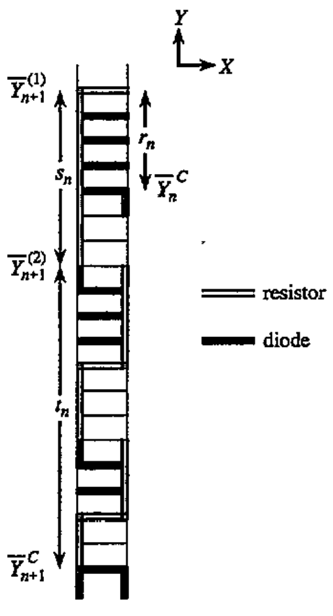

Figure 7. An example of the bond configuration on $v_{-n} \cup h_{-(n+1)} \cup v_{-(n+1)}$, where thin bonds are either resistors or diodes. By definition, the vertical bonds just below $\bar{Y}^C_n$, $\bar{Y}^{(2)}_{n+1}$ and $\bar{Y}^C_{n+1}$ should be diodes.

It is easy to obtain

$$P(r,s)=\begin{cases} p(1-p-2q)q^{s-1}(p+q)^{r-s} & \text{if} \quad r\geqslant s\geqslant 1 \\ q(p+q)^r & \text{if} \quad r\geqslant 0, s=0 \\ (1-p-2q)(p+q)q^r\{1-(p+q)\}^{s-r-1} & \text{if} \quad s>r\geqslant 0\,. \end{cases} \tag{4.24}$$

Therefore the first two terms of the RHS of (4.22) are given as

$$E[r_n]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} rP(r,s)=\frac{p+q}{1-(p+q)} \tag{4.25}$$

and

$$E[s_n]=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} sP(r,s)=\frac{\{1-(p+2q)\}[1-(p+q)\{1-(p+q)\}]}{(p+q)\{1-(p+q)\}(1-q)}\,. \tag{4.26}$$

For the conditional probability $P(t|r,s,\tilde{r},\tilde{s})$ we observe the following facts. (i) If $r\geqslant s$, $P(t|r,s,\tilde{r},\tilde{s})=\delta_{t,0}$. The reason is that all the horizontal bonds on $h_{-(n+1)}$ whose ordinates are between $\bar{Y}^{(2)}_{n+1}$ and $\bar{Y}^{C}_{n}$ are diodes for this case and the vertical bond just below $\bar{Y}^{C}_{n}$ is a diode by definition. (ii) If $r_n<s_n$ and $\tilde{r}_n<\tilde{s}_n$, or if $s_n-r_n>\tilde{r}_n-\tilde{s}_n\geqslant 0$, we have $\bar{Y}^{(2)}_{n+1}<\bar{Y}^{C}_{n}$ and $\bar{Y}^{(2)}_{n+1}<\bar{Y}^{C}_{n-1}$. Therefore $P(t|r,s,\tilde{r},\tilde{s})$ is independent of $r,s,\tilde{r},\tilde{s}$ and given by a function of t, which we write f_t. (iii) If $\tilde{r}_n-\tilde{s}_n\geqslant s_n-r_n>0$, $\bar{Y}^{(2)}_{n+1}<\bar{Y}^{C}_{n}$ but $\bar{Y}^{(2)}_{n+1}\geqslant\bar{Y}^{C}_{n-1}$, since $\bar{Y}^{(2)}_{n+1}-\bar{Y}^{C}_{n-1}=(\tilde{r}_n-\tilde{s}_n)-(s_n-r_n)$. In this case $P(t|r,s,\tilde{r},\tilde{s})$ is a function of t and $(\tilde{r}-\tilde{s})-(s-r)$ and we write $P(t|r,s,\tilde{r},\tilde{s})=g(t,(\tilde{r}-\tilde{s})-(s-r))$. By this observation, we obtain (see appendix A.1 for details)

$$E[t_n]\doteq\left\{P(r<s)^2+P(s-r>\tilde{r}-\tilde{s}\geqslant 0)\right\}[t]_f+P(\tilde{r}-\tilde{s}\geqslant s-r>0)\langle[t]_g\rangle \tag{4.27}$$

where

$$[t]_f=\sum_{t=0}^{\infty} tf_t \tag{4.28}$$

and

$$\langle[t]_g\rangle=\frac{\displaystyle\sum_{h=0}^{\infty}(p+q)^h[t]_g(h)}{\displaystyle\sum_{h=0}^{\infty}(p+q)^h} \tag{4.29}$$

with

$$[t]_g(h)=\sum_{t=0}^{\infty} tg(t,h)\,. \tag{4.30}$$

It is easy to obtain the following probabilities from (4.23) and (4.24):

$$\begin{aligned} P(r<s)^2&+P(s-r>\tilde{r}-\tilde{s}\geqslant 0)\\ &=\frac{[1-(p+2q)][1-(p+2q)[1-(p+q)\{1-(p+q)\}]]}{(1-q)^2[1-(p+q)\{1-(p+q)\}]} \end{aligned} \tag{4.31}$$

and

$$P(\tilde{r}-\tilde{s}>s-r>0)=\frac{[1-(p+2q)](p+q)^3}{(1-q)^2[1-(p+q)\{1-(p+q)\}]}\,. \tag{4.32}$$

Appendices A.2 and A.3 are devoted to derive the following results:

$$[t]_f = \frac{\{1-(p+q)\}\big[\{1-(p+2q)\}+(p+q)(p+2q)\{1-(p+q)\}\big]}{(p+q)^2[1-(p+q)\{1-(p+q)\}]} \tag{4.33}$$

and

$$\langle [t]_g \rangle = [t]_f - \frac{\{-p+(p+q)^2\}\{1-(p+q)\}\sum_{n=0}^{4} a_n(q)p^n}{(p+q)^3\{1-q(p+q)\}\big[1-(1-p)(p+q)-q^2\{1-(p+q)\}\big]} \tag{4.34}$$

with

$$\begin{aligned}
a_0(q) &= 2q^7 - 3q^6 - q^5 + 2q^4 - 2q^3 + q^2 + q - 1 \\
a_1(q) &= q(7q^5 - 9q^4 - 5q^3 + 5q^2 - 1) \\
a_2(q) &= 9q^5 - 10q^4 - 6q^3 + 4q^2 + q - 1 \\
a_3(q) &= q(5q^3 - 5q^2 - 2q + 1) \\
a_4(q) &= q^2(q-1).
\end{aligned}$$

Substituting (4.31)–(4.34) for (4.27) gives $E[t_n]$.

4.3. *Lower bound for $p_c(q)$*

We obtain the following lower bound for the critical line $p = p_c(q)$ on the (p, q)-plane.

Proposition 4.2. Let

$$E[W^C] = E[r_n] - E[s_n] - E[t_n]. \tag{4.35}$$

Define for $0 \leqslant q \leqslant \frac{1}{2}$,

$$p_L(q) = \sup\{p : 0 \leqslant p \leqslant 1-2q \text{ and } E[W^C] < 1\}. \tag{4.36}$$

Then

$$p_L(q) \leqslant p_c(q) \qquad \text{for} \quad 0 \leqslant q \leqslant \tfrac{1}{2}. \tag{4.37}$$

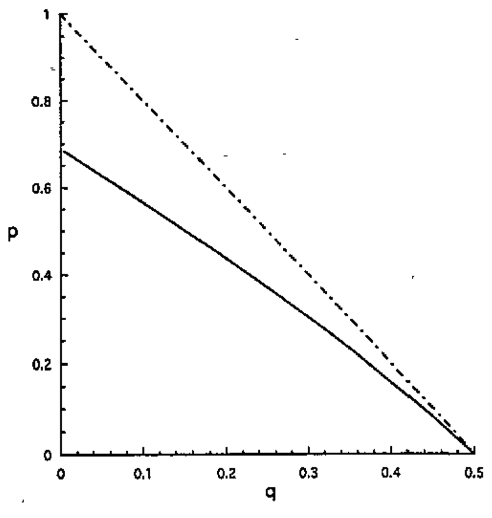

Figure 8. The line denotes the lower bound $p = p_L(q)$ given by proposition 4.2. The critical line $p = p_c(q)$ should exist above this line.

In other words, for $0 \leqslant q \leqslant \frac{1}{2}$, if $0 \leqslant p < p_{\mathrm{L}}(q)$, then both of the processes $\tilde{\eta}_t^{\{0\}}$ and $\tilde{\eta}_t^{\mathbf{Z}_-}$ become extinct with probability one.

Figure 8 shows the numerical values of the lower bound $p = p_{\mathrm{L}}(q)$.

5. Phase diagram on the (p_1, p_2)-plane

5.1. Lower bound for the critical line

The lower bound for $p = p_c(q)$ given by proposition 4.2 is mapped to the bound for the critical line in the (p_1, p_2)-plane by (2.6). We give here the rather lengthy but explicit expression of the result.

Theorem 5.1. Let

$$\begin{aligned} C_{\mathrm{L}}(p_1, p_2) = &\{1 - p_1(p_2 - p_1)\}\{1 - (p_2 - p_1)\} \\ &\times[1 - \{1 - (2p_1 - p_2)\}p_1 - (p_2 - p_1)^2(1 - p_1)]\ B_1(p_1, p_2) \\ &+\{p_2 - p_1(2 - p_1)\}p_1^2(p_2 - 1)(p_1 - 1)^2\ B_2(p_1, p_2) \end{aligned} \tag{5.1}$$

with

$$\begin{aligned} B_1(p_1, p_2) = &\ 2p_1^6 - (p_2 + 2)p_1^5 - (p_2^2 - 2p_2 - 2)p_1^4 + (3p_2^2 - 3p_2 - 1)p_1^3 \\ &-4p_2(p_2 - 1)p_1^2 + (3p_2 - 1)(p_2 - 1)p_1 - (p_2 - 1)^2 \\ B_2(p_1, p_2) = &\ (p_2 + 1)p_1^6 - (3p_2^2 + 3p_2 - 1)p_1^5 + p_2^2(3p_2 + 4)p_1^4 \\ &-(p_2^4 + 3p_2^3 + 4p_2^2 - p_2 - 2)p_1^3 + (p_2^4 + 4p_2^3 - p_2^2 - 2p_2 + 1)p_1^2 \\ &-(p_2 - 1)(p_2^3 + p_2^2 + 2p_2 + 1)p_1 + (p_2 + 1)(p_2 - 1)^2 . \end{aligned} \tag{5.2}$$

In the parameter region where

$$0 \leqslant p_1 \leqslant 1 \qquad 0 \leqslant p_2 \leqslant 1 \qquad p_1 \leqslant p_2 \leqslant 2p_1 \tag{5.3}$$

and

$$C_{\mathrm{L}}(p_1, p_2) < 0 \tag{5.4}$$

$$\lim_{t\to\infty} P(\eta_t^{\mathbf{Z}_e} \neq \emptyset) = 0 . \tag{5.5}$$

This implies that the process η_t^A starting from any initial state A dies out with probability one.

In figures 9(*a*) and (*b*), we show the line $C_{\mathrm{L}}(p_1, p_2) = 0$ in the region (5.3). Theorem 5.1 says that the region to the left of this line is completely included in the extinction phase and that the critical line separating this phase and the survival phase should lie to the right of this line. It should be noted that in the regions $p_2 > 2p_1$ and $p_2 < p_1$ the percolation substructure explained in section 2 loses its meaning, since $p = 2p_1 - p_2 < 0$ and $q = p_2 - p_1 < 0$, respectively, by the relation (2.6).

5.2. Upper bound for the critical line by Liggett

Recently Liggett (1994) proved the following remarkable result.

Theorem 5.2 (Liggett). In the parameter region where

$$\tfrac{1}{2} \leqslant p_1 \leqslant 1 \qquad p_1 \leqslant p_2 \leqslant 1 \tag{5.6}$$

and

$$C_{\mathrm{U}}(p_1, p_2) = p_2 - 4p_1(1 - p_1) \geqslant 0 \tag{5.7}$$

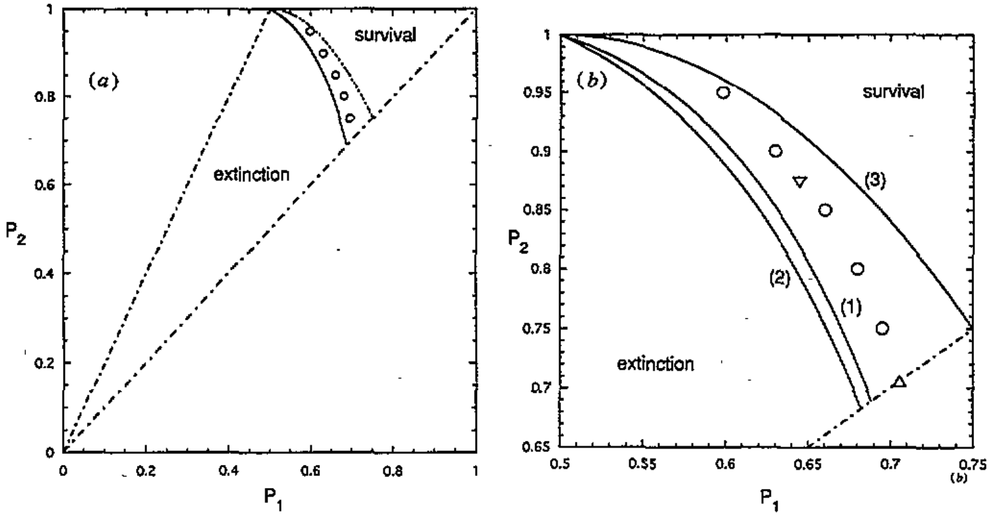

Figure 9. (*a*) The full curve is the lower bound $p_1 = p_{1\mathrm{L}}(p_2)$ for the critical line given by theorem 5.1. The dotted line is the upper bound given by Liggett (theorem 5.2). The critical values estimated by the Monte Carlo simulation are marked by $\circ$'s. (*b*) An enlarged figure. Rigorous bounds for the critical line are summarized: (1) the lower bound given by theorem 5.1, (2) the simplified lower bound given by theorem 5.7, and (3) Liggett's upper bound given by theorem 5.2. $\circ$'s denote the estimated critical values by the Monte Carlo simulation. The point marked by Δ (resp. ∇) denotes the value α_c (resp. β_c) evaluated by Onody and Neves (1992) (resp. Baxter and Guttmann 1988).

$$P(\eta_t^{\mathbf{Z}_e} \neq \emptyset \text{ for all } t \geqslant 0) > 0 \tag{5.8}$$

that is, the process survives with a positive probability.

We also show in figure 9 the line $C_{\mathrm{U}}(p_1, p_2) = 0$ in the region (5.6). The proof of theorem 5.2 is based on the method called the *Holley–Liggett argument*. This argument was first applied by Holley and Liggett (1978) to the basic contact process, which can be regarded as the continuous-time version of the bond-directed percolation model. After that this technique has been extended by Liggett (1991a, 1991b) and by Katori and Konno (1993) for the modified and the generalized contact processes. Liggett (1995) also reported the improvement of the Holley–Liggett result for the basic contact process. Liggett's new result (1994) first showed that this method is also applicable to discrete-time processes.

Theorems 5.1 and 5.2 imply that the true critical line exists between the two lines $C_{\mathrm{L}}(p_1, p_2) = 0$ and $C_{\mathrm{U}}(p_1, p_2) = 0$ in the region (5.3). Let $p_1 = p_{1\mathrm{c}}(p_2)$ denote the critical line and

$$p_{1\mathrm{L}}(p_2) = \sup\left\{p_1 : \tfrac{1}{2}p_2 \leqslant p_1 \leqslant p_2 \text{ and } C_{\mathrm{L}}(p_1, p_2) < 0\right\} \tag{5.9}$$

$$p_{1\mathrm{U}}(p_2) = \inf\left\{p_1 : \tfrac{1}{2}p_2 \leqslant p_1 \leqslant p_2 \text{ and } C_{\mathrm{U}}(p_1, p_2) > 0\right\}. \tag{5.10}$$

Then we have the following result.

Theorem 5.3. There exists a critical line $p_1 = p_{1\mathrm{c}}(p_2)$ in the region (5.3) and

$$p_{1\mathrm{L}}(p_2) \leqslant p_{1\mathrm{c}}(p_2) \leqslant p_{1\mathrm{U}}(p_2). \tag{5.11}$$

5.3. Special cases

Bond-directed percolation with bond concentration β. If we put p_1 and p_2 as (2.9), $C_L(p_1, p_2)$ may be factorized as

$$C_L(\beta, \beta(2-\beta)) = (\beta^2-\beta+1)(\beta^3-\beta^2+1)(\beta^5-3\beta^4+4\beta^3-\beta^2-\beta+1) \times(-\beta^8+8\beta^7-22\beta^6+36\beta^5-37\beta^4+27\beta^3-14\beta^2+5\beta-1) . \tag{5.12}$$

Let

$$C_L^{bond}(\beta) = -\beta^8+8\beta^7-22\beta^6+36\beta^5-37\beta^4+27\beta^3-14\beta^2+5\beta-1 \tag{5.13}$$

and

$$\beta_L = \sup\{\beta : C_L^{bond}(\beta) < 0\} . \tag{5.14}$$

Then

$$\beta_L = 0.626\,121\ldots \qquad \text{and} \qquad \beta_L \leqslant \beta_c \tag{5.15}$$

for the critical value β_c of the bond-directed percolation. This lower bound was reported in Dhar (1982) as the result of his second strategy. The most reliable value estimated so far is

$$\beta_c = 0.644\,701 \pm 0.000\,001 \tag{5.16}$$

given by Baxter and Guttmann (1988). Liggett's upper bound (theorem 5.2) gives for this case

$$\beta_c \leqslant \tfrac{2}{3} . \tag{5.17}$$

Site-directed percolation with site concentration α. If we set the parameters as (2.8), we have

$$C_L^{site}(\alpha) = C_L(\alpha, \alpha) = \alpha^9-4\alpha^8+10\alpha^7-16\alpha^6+21\alpha^5-20\alpha^4+15\alpha^3-9\alpha^2+4\alpha-1 . \tag{5.18}$$

It gives the lower bound for the critical value α_c of site-directed percolation,

$$\alpha_L \leqslant \alpha_c \tag{5.19}$$

with

$$\alpha_L = \sup\{\alpha : C_L^{site}(\alpha) < 0\} = 0.688\,547\ldots . \tag{5.20}$$

This lower bound for α_c is new one and slightly better than the bound by Gray *et al* (1980). The most reliable value of α_c estimated so far is

$$\alpha_c = 0.705\,489 \pm 0.000\,004 \tag{5.21}$$

given by Onody and Neves (1992). Liggett's upper bound gives for this case

$$\alpha_c \leqslant \tfrac{3}{4} . \tag{5.22}$$

The limit $p_2 \to 1$ and its vicinity. It should be remarked that

$$\lim_{p_2\to 1} C_L(p_1, p_2) = p_1^5(p_1^2-p_1+1)^3(2p_1-1) \tag{5.23}$$

and

$$\lim_{p_2\to 1} C_U(p_1, p_2) = (2p_1-1)^2 . \tag{5.24}$$

Therefore we have the following corollary of theorem 5.3.

Corollary 5.4.

$$\lim_{p_2 \to 1} p_{1c}(p_2) = \tfrac{1}{2}\,. \tag{5.25}$$

The next question is the convergence rate of $p_{1c}(p_2)$ to $\frac{1}{2}$ as $p_2 \to 1$. Durrett (1988) proposed the following conjecture.

Conjecture 5.5 (Durrett). Assume that $0 < 1 - p_2 \ll 1$. Then there exists an exponent θ such that

$$p_{1c}(p_2) - \tfrac{1}{2} \simeq C(1 - p_2)^{\theta} \tag{5.26}$$

where $\simeq$ means that if the constant C is taken to be sufficiently large (resp. small) the RHS gives the upper (resp. lower) bound of the LHS.

Since we observe for $0 < 1 - p_2 \ll 1$,

$$p_{1L}(p_2) - \tfrac{1}{2} = \tfrac{113}{54}(1 - p_2) + \mathcal{O}((1 - p_2)^2) \tag{5.27}$$

and

$$p_{1U}(p_2) - \tfrac{1}{2} = \tfrac{1}{2}(1 - p_2)^{1/2} \tag{5.28}$$

we obtain the following result.

Corollary 5.6. If conjecture 5.5 is correct, then

$$\tfrac{1}{2} \leqslant \theta \leqslant 1\,. \tag{5.29}$$

5.4. *Simplified version*

The function $C_L(p_1, p_2)$ which gives the lower bound for the critical line is rather complicated. The reason is that $E[t_n]$ is lengthy. By definition, $E[t_n] \geqslant 0$ and thus we have

$$E[W^C] \leqslant E[r_n] - E[s_n]\,. \tag{5.30}$$

Therefore if $E[r_n] - E[s_n] < 1$, then $E[W^C] < 1$ and it follows lemma 4.1 that the process becomes extinct. This observation leads a slightly worse, but much more simply expressed lower bound for the critical line $p_1 = p_{1c}(p_2)$.

Theorem 5.7. Let

$$\tilde{C}_L(p_1, p_2) = p_2 - \frac{1 - 2p_1^3}{(1 - p_1)(1 + p_1)} \tag{5.31}$$

and

$$\tilde{p}_{1L}(p_2) = \sup\{p_1 : \frac{p_2}{2} \leqslant p_1 \leqslant p_2 \text{ and } \tilde{C}_L(p_1, p_2) < 0\}\,. \tag{5.32}$$

Then

$$\tilde{p}_{1L}(p_2) \leqslant p_{1c}(p_2)\,. \tag{5.33}$$

It should be remarked that for bond-directed percolation this simplified version gives $\tilde{\beta}_L \leqslant \beta_c$ with $\tilde{\beta}_L = (-1+\sqrt{5})/2 = 0.618\,033\ldots$, and for site-directed percolation, $\tilde{\alpha}_L \leqslant \alpha_c$ with $\tilde{\alpha}_L = \{\frac{1}{2} - \sqrt{93}/18\}^{1/3} + \{\frac{1}{2} + \sqrt{93}/18\}^{1/3} = 0.682\,327\ldots$ as special cases. The former bound $\tilde{\beta}_L$ is the same as the value reported in Dhar (1982) as the result of his first strategy. This simplified lower bound is also shown in figure 9(b).

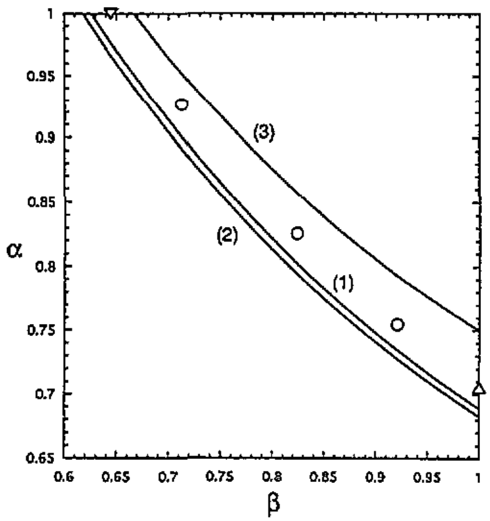

Figure 10. Bounds for the critical line of the mixed site–bond directed percolation: (1) the lower bound given by theorem 5.1, (2) the simplified lower bound given by theorem 5.7, and (3) Liggett's upper bound given by theorem 5.2. The critical values estimated by the Monte Carlo simulation are denoted by o's. The point marked by Δ (resp. ∇) denotes the value α_c (resp. β_c) evaluated by Onody and Neves (1992) (resp. Baxter and Guttmann 1988).

5.5. *Computer simulation*

We have performed the Monte Carlo simulation for the present process η_t. The estimated critical values are plotted in figures 9(a) and (b). These results fall between our lower bound and the upper bound of Liggett as expected. More details will be reported in Tretyakov *et al* (1995).

5.6. *Mixed site–bond directed percolation*

If we parameterize p_1 and p_2 as (2.10), the present process is identified with the mixed site–bond directed percolation with site concentration α and bond concentration β. We can consider the (α, β) phase diagram on which there exists a critical line between the percolation phase and the non-percolation phase. We show the numerical values of the lower bound given by theorem 5.1 in figure 10. The simplified version, theorem 5.7, gives the line

$$\alpha^3\beta^4 - \alpha\beta^2 + 2\alpha\beta = 1 \tag{5.34}$$

which also gives the lower bound. We show this line as well as the upper bound of Liggett in figure 10. Quite recently, Tretyakov and Inui (1995) reported a precise study by simulations. They have compared their results with our lower bound (5.34).

5.7. *Remarks on future problems*

In the present paper, we have reported rigorous results on the phase diagram of two-neighbour SCA. Both our theorem and Liggett's theorem are based on the attractiveness of the processes; the bounds are only valid for the parameter region $p_2 \geqslant p_1$. We find, however, that by comparing with the simulation results the line $p_1 = p_{1\mathrm{L}}(p_2)$ seems to give a good 'lower bound' for the true critical line even for the region $p_2 < p_1$, which will be reported in the forthcoming paper (Tretyakov *et al* 1995).

The two-neighbour SCA can be regarded as the discrete-time version of the generalized contact process which has a parameter θ and we simply call it the θ-contact process. For $1 \leqslant \theta \leqslant 2$ rigorous lower and upper bounds have been given (Katori and Konno 1993) and then the lower bound was extended for $\theta > 2$ (Katori 1994), but we have had no rigorous bounds for the non-attractive cases, $0 \leqslant \theta < 1$. Jensen and Dickman (1994) extensively studied the θ-contact process by the series expansion method and showed that the line given by Katori and Konno (1993) seems to bound the critical line estimated by them also in the region $0 \leqslant \theta < 1$. Further study of non-attractive systems are required both for continuous-time processes and SCA.

Giving the precise definition of the angle ϕ, we have clarified the foundation of Dhar's method and extended his results. In order to obtain lower bounds for critical values, we introduced approximate processes by making the appropriate restriction for the paths on the spatio-temporal plane; we replace the set B_{-1} by its subset C as explained in subsection 4.1. In this paper we have chosen the set C as a strip consisting of two successive columns and the horizontal bonds between them. It should be remarked again that even though C is a rather narrow region, the difference between our lower bounds and the critical values estimated by series expansion methods or simulations is only a few per cent. It can be proved that if we enlarge the set C and let $C \to B_{-1}$, the obtained lower bounds converge to the true values (Wierman 1983). The present results imply that the convergence of Dhar's method is excellent in comparison with other methods giving lower bounds (see Konno 1994). Applications of Dhar's method to the percolation models on a triangular lattice will be reported elsewhere (Tsukahara and Katori 1995).

As explained in section 3, Dhar's method can be regarded as a kind of contour method. One of the remarkable features of Dhar's method is that the overhang structures of the contour can be taken into account effectively and systematically. Application of Dhar's method to the continuous-time interacting particle systems may be an important problem in the future.

Acknowledgments

The present authors would like to thank Norio Konno for teaching us Liggett's new results. They also thank Alex Yu. Tretyakov and Norio Inui for valuable comments and for sending us their paper prior to publication. They are grateful to Ronald Dickman for useful discussions and for a careful reading of the manuscript. This work is partially financed by the Scientific Research Fund of the Ministry of Education, Science and Culture (Japan).

Appendix

Appendix A.1. Derivation of (4.27)

The indicator function $1_{\{\omega\}}$ is defined as $1_{\{\omega\}} = 1$ if the event ω occurs and $1_{\{\omega\}} = 0$ otherwise. By equations (4.21), (4.23) and the facts mentioned above (4.27), we have

$$\begin{aligned} E[t_n] &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\tilde{r}=0}^{\infty}\sum_{\tilde{s}=0}^{\infty}\big(1_{\{r\geqslant s\}} + 1_{\{r<s\}}1_{\{\tilde{r}<\tilde{s}\}} \\ &\qquad + 1_{\{s-r>\tilde{r}-\tilde{s}\geqslant 0\}} + 1_{\{\tilde{r}-\tilde{s}\geqslant s-r>0\}}\big)\ tP(r,s,t,\tilde{r},\tilde{s}) \\ &= \left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}1_{\{r<s\}}P(r,s)\right)\left(\sum_{\tilde{r}=0}^{\infty}\sum_{\tilde{s}=0}^{\infty}1_{\{\tilde{r}<\tilde{s}\}}P(\tilde{r},\tilde{s})\right)\sum_{t=0}^{\infty}tf_t \end{aligned}$$

$$+\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\tilde{r}=0}^{\infty}\sum_{\tilde{s}=0}^{\infty}1_{\{s-r>\tilde{r}-\tilde{s}\geqslant 0\}}P(r,s)P(\tilde{r},\tilde{s})\sum_{t=0}^{\infty}tf_t$$

$$+\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\tilde{r}=0}^{\infty}\sum_{\tilde{s}=0}^{\infty}\sum_{t=0}^{\infty}1_{\{\tilde{r}-\tilde{s}\geqslant s-r>0\}}P(r,s)P(\tilde{r},\tilde{s})tg(t,(\tilde{r}-\tilde{s})-(s-r)) \quad (A.1)$$

Let S_1 be the last term in the RHS of the second equality in (A.1). Then

$$S_1=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\tilde{r}=0}^{\infty}\sum_{\tilde{s}=0}^{\infty}1_{\{\tilde{r}-\tilde{s}>0\}}1_{\{s-r>0\}}1_{\{(\tilde{r}-\tilde{s})-(s-r)\geqslant 0\}}$$
$$\times P(\tilde{r},\tilde{s})P(r,s)H((\tilde{r}-\tilde{s})-(s-r)) \quad (A.2)$$

with

$$H((\tilde{r}-\tilde{s})-(s-r))=\sum_{t=0}^{\infty}tg(t,(\tilde{r}-\tilde{s})-(s-r)). \quad (A.3)$$

It is rewritten as follows:

$$S_1=\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}P(r,s)\sum_{\tilde{s}=0}^{\infty}\sum_{\tilde{r}=\tilde{s}+1}^{\infty}P(\tilde{r},\tilde{s})1_{\{(\tilde{r}-\tilde{s})\geqslant(s-r)\}}H((\tilde{r}-\tilde{s})-(s-r))$$
$$=\sum_{r=0}^{\infty}\sum_{\alpha=1}^{\infty}P(r,\alpha+r)\sum_{\tilde{s}=0}^{\infty}\sum_{h=0}^{\infty}P(\alpha+\tilde{s}+h,\tilde{s})H(h). \quad (A.4)$$

Since $P(r,s)$ is given as (4.24), we find that

$$\sum_{h=0}^{\infty}P(\alpha+\tilde{s}+h,\tilde{s})H(h)=P(\alpha+\tilde{s},\tilde{s})\sum_{h=0}^{\infty}(p+q)^hH(h)$$
$$=\frac{\sum_{\tilde{r}=\alpha+\tilde{s}}^{\infty}P(\tilde{r},\tilde{s})}{\sum_{h=0}^{\infty}(p+q)^h}\sum_{h=0}^{\infty}(p+q)^hH(h)$$
$$=\sum_{\tilde{r}=\alpha+\tilde{s}}^{\infty}P(\tilde{r},\tilde{s})\ \langle H\rangle \quad (A.5)$$

where we have used the notation (4.29). By equations (A.3)–(A.5) and using the notation (4.30), we have

$$S_1=\sum_{r=0}^{\infty}\sum_{s=r+1}^{\infty}P(r,s)\sum_{\tilde{s}=0}^{\infty}\sum_{\tilde{r}=(s-r)+\tilde{s}}^{\infty}P(\tilde{r},\tilde{s})\langle[t]_g\rangle. \quad (A.6)$$

The equation (4.27) follows (A.1) with (A.6).

Appendix A.2. Derivation of (4.33)

It is easy to obtain the following equation for f_t with $t\geqslant 1$, which is a simple generalization of Dhar's equation for the bond-directed percolation (see equation (A8) in Dhar 1982).

$$f_t=\frac{q}{p+q}\sum_{a=1}^{\infty}[(p+q)\{1-(p+q)\}]^a\sum_{c=0}^{\infty}f_c\delta_{t,a+c}$$

$$+\frac{1-(p+2q)}{1-(p+q)}\sum_{a=1}^{\infty}\left[(p+q)\{1-(p+q)\}\right]^a$$
$$\times\sum_{b=1}^{\infty}\{1-(p+q)\}^b\sum_{c=0}^{\infty}f_c\delta_{t,a+b+c} \tag{A.7}$$

for $t \geqslant 1$. When $t=0$, we have

$$f_0=\sum_{a'=0}^{\infty}(p+q)\left[(p+q)\{1-(p+q)\}\right]^{a'}$$
$$=\frac{p+q}{1-(p+q)\{1-(p+q)\}}. \tag{A.8}$$

Following Dhar (1982), we introduce the generating function of f_t as

$$F(x)=\sum_{t=0}^{\infty}f_t x^t \tag{A.9}$$

and (A.7) is solved as follows:

$$F(x)=\frac{f_0}{1-W(x)} \tag{A.10}$$

with

$$W(x)=\frac{q\{1-(p+q)\}x}{1-(p+q)\{1-(p+q)\}x}$$
$$+\frac{(p+q)\{1-(p+2q)\}\{1-(p+q)\}x^2}{[1-\{1-(p+q)\}x][1-(p+q)\{1-(p+q)\}x]}. \tag{A.11}$$

As expected we find that

$$F(1)=\sum_{t=0}^{\infty}f_t=1. \tag{A.12}$$

The result (4.33) is obtained by the formula

$$[t]_f=\left.\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right|_{x=1}. \tag{A.13}$$

Appendix A.3. Derivation of (4.34)

It is straightforward to obtain the following equations for $g(t,h)$:

$$g(t,h)=\sum_{a=h+1}^{\infty}\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^{h+1}\{1-(p+q)\}^{a-h-1}\frac{q}{p+q}(p+q)^a f_c\delta_{c,a+c}$$
$$+\sum_{a=h+1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^{h+1}\{1-(p+q)\}^{a-h-1}$$
$$\times\frac{1-(p+2q)}{1-(p+q)}(p+q)^a\{1-(p+q)\}^b f_c\delta_{t,a+b+c}$$
$$+\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^{h}\frac{q}{p+q}(p+q)^h f_c\delta_{t,h+c}$$
$$+\sum_{b=1}^{\infty}\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^{h}\frac{1-(p+2q)}{1-(p+q)}(p+q)^h\{1-(p+q)\}^b f_c\delta_{t,h+b+c}$$

$$+1_{\{h\geqslant 2\}}\sum_{a=1}^{h-1}\sum_{b=h-a+1}^{\infty}\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^a\frac{1-(p+2q)}{1-(p+q)}(p+q)^a\{1-(p+q)\}^b$$
$$\times f_c\delta_{t,a+b+c}$$
$$+1_{\{h\geqslant 2\}}\sum_{a=1}^{h-1}\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^a\frac{q}{p+q}(p+q)^a g(c,h-a)\delta_{t,a+c}$$
$$+1_{\{h\geqslant 2\}}\sum_{a=1}^{h-1}\sum_{b=1}^{h-a}\sum_{c=0}^{\infty}\left(\frac{q}{p+q}\right)^a\frac{1-(p+2q)}{1-(p+q)}(p+q)^a\{1-(p+q)\}^b$$
$$\times g(c,h-(a+b))\delta_{t,a+b+c}$$
$$\text{for}\quad t\geqslant 1\quad h\geqslant 1 \tag{A.14}$$

$$g(0,h)=\frac{p}{(p+q)(1-q)}+\frac{[-p+(p+q)^2]\{1-(p+q)\}}{(p+q)(1-q)[1-(p+q)\{1-(p+q)\}]}q^{h+1}$$
$$\text{for}\quad h\geqslant 1 \tag{A.15}$$

$$g(t,0)=\frac{q}{(p+q)\{1-(p+q)\}}f_t\qquad\text{for}\quad t\geqslant 1 \tag{A.16}$$

and

$$g(0,0)=\frac{p+(p+q)[-p+(p+q)^2]}{(p+q)[1-(p+q)\{1-(p+q)\}]}. \tag{A.17}$$

In order to solve these equations, here we introduce two kinds of generating functions for $g(t,h)$. The first one is

$$G(x,h)=\sum_{t=0}^{\infty}x^t g(t,h) \tag{A.18}$$

and the second one is

$$\tilde{G}(x,p+q)=\sum_{h=0}^{\infty}(p+q)^h G(x,h). \tag{A.19}$$

Then we have the equation in the following form:

$$A(p,q)\tilde{G}(x,p+q)=B(p,q)G(x,0)+C(p,q)+D(p,q)F(x). \tag{A.20}$$

It is easy to see that

$$G(x,0)=-\frac{-p+(p+q)^2}{(p+q)\{1-(p+q)\}}+\frac{q}{(p+q)\{1-(p+q)\}}F(x). \tag{A.21}$$

The coefficients $A(p,q)$, $B(p,q)$, $C(p,q)$ and $D(p,q)$ are functions of p and q. Since they are rather lengthy, we omit them here. The result (4.34) is derived by the formula.

$$\langle [t]_g\rangle=\{1-(p+q)\}\frac{\mathrm{d}\tilde{G}(x,p+q)}{\mathrm{d}x}\bigg|_{x=1}. \tag{A.22}$$

References

Balister P, Bollabás B and Stacey A 1994 Improved upper bounds for the critical probability of oriented percolation in two dimensions *Random Structure and Algorithms*

Baxter R J and Guttmann A J 1988 Series expansion of the percolation probability for the directed square lattice *J. Phys. A: Math. Gen.* **21** 3193–204

Dhar D 1982 Diode-resistor percolation in two and three dimensions: I. Upper bounds on critical probability *J. Phys. A: Math. Gen.* **15** 1849–58

Dhar D, Barma M and Phani M K 1981 Duality transformations for two-dimensional directed percolation and resistance problems *Phys. Rev. Lett.* **47** 1238–41

Dickman R 1993 Nonequilibrium phase transitions in catalysis and population models *Int. J. Mod. Phys.* C **4** 271–7

Durrett R 1984 Oriented percolation in two dimensions *Ann. Probab.* **12** 999–1040

Durrett R 1988 *Lecture Notes on Particle Systems and Percolation* (Pacific Grove, CA: Wadsworth and Brooks/Cole)

Gray L, Wierman J C and Smythe R T 1980 Lower bounds for the critical probability in percolation models with oriented bonds *J. App. Prob.* **17** 979–86

Harris T E 1974 Contact interactions on a lattice *Ann. Probab.* **2** 969–88

Holley R and Liggett T M 1978 The survival of contact processes *Ann. Probab.* **6** 198–206

Inui N, Katori M and Uzawa T 1995 Duality and universality in non-equilibrium lattice models *J. Phys. A: Math. Gen.* **28** 1817–30

Jensen I and Dickman R 1994 Series analysis of the generalized contact process *Physica* **203A** 175–88

Katori M 1994 Reformulation of Gray's duality for attractive spin systems and its applications *J. Phys. A: Math. Gen.* **27** 3191–211

Katori M and Konno N 1993 Bounds for the critical line of the θ-contact processes with $1 \leqslant \theta \leqslant 2$ *J. Phys. A: Math. Gen.* **26** 6597–614

Kinzel W 1985 Phase transitions of cellular automata *Z. Phys.* B **58** 229–44 and references therein

Konno N 1994 *Phase Transitions of Interacting Particle Systems* (Singapore: World Scientific)

Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)

——1991a Spatially inhomogeneous contact process *Spatial Stochastic Processes* ed K S Alexander and J C Watkins (Boston, MA: Birkhäuser) pp 105–40

——1991b The periodic threshold contact process *Random Walks, Brownian Motion and Interacting Particle Systems* ed R Durrett and H Kesten (Boston, MA: Birkhäuser) pp 339–58

——1994 Survival of discrete time growth models, with applications to oriented percolation *Preprint* (to appear in *Ann. Appl. Probab.*)

——1995 Improved upper bounds for the contact process critical value *Ann. Probab.* **23**

Onody R N and Neves U P C 1992 Series expansion of the directed percolation probability *J. Phys. A: Math. Gen.* **25** 6609–15

Tretyakov A Yu and Inui N 1995 Critical behaviour for mixed site-bond directed percolation *J. Phys. A: Math. Gen.* **28** 3985

Tretyakov A Yu, Inui N, Tsukahara H and Katori M 1995 in preparation

Tsukahara H and Katori M 1995 in preparation

Wierman J C 1983 On square lattice directed percolation and resistance models *J. Phys. A: Math. Gen.* **16** 3545–51